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Note on ordering and complexity of Platonic and Archimedean polyhedra based on solid angles

Damir Vukičević · Alexandru T. Balaban

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Abstract The missing values for the solid angles of the two snub semiregular polyhedra have been calculated, and integrated into the whole series of Platonic and Archimedean polyhedra. This is the only criterion which so far gives an unambiguous answer (without any degeneracy leading to posets) on how to order these polyhedra according to their increasing complexity.

Keywords Regular (Platonic) polyhedra · Semiregular (Archimedean) polyhedra · Complexity · Ordering · Solid angles

1 Introduction

Some of the most interesting hydrocarbons have carbon scaffolds that are Platonic (regular) polyhedra which have congruent faces that are regular polygons. The synthesis of *tetrahedrane* and its stable substituted derivative (G. Maier) (tetra-*tert*-butyltetrahedrane), *cubane*, whose hepta- and octa-nitro-derivatives are "energetic materials" (P. E. Eaton), and *dodecahedrane* (L. A. Paquette, then H. Prinzbach) were challenges that required much skill and inventivity. These three "trivalent regular polyhedra" $(CH)_{2k}$ are valence isomers of annulenes. Relevant references are found in the reference section [1–3].

Archimedean (semiregular) polyhedra have faces that are regular polygons of more than one type. The trivalent Archimedean (semiregular) polyhedra (Nos. **6**, **8**, **9**, **13–15**)

D. Vukičević
Department of Mathematics, University of Split, Nikole Tesle 12, 21000 Split, Croatia
e-mail: vukicevi@pmfst.hr

A. T. Balaban (✉)
Texas A&M University at Galveston, 5007 Avenue U, Galveston, TX 77551, USA
e-mail: balabana@tamug.edu

in Table 1 are also valence isomers of annulenes and they provide further challenges for the future. The most famous of these is buckminsterfullerene, C_{60} (**15**), the discovery of which was honored with the Nobel Prize for Chemistry awarded in 1996 to Robert F. Curl Jr., Harry W. Kroto, and Richard E. Smalley [4,5]. The large-scale preparation of fullerenes was initiated by W. Krätschmer and D. R. Huffman [6]. More recently, Lawrence T. Scott during the last 10 years gradually developed rational synthetic methods for obtaining "geodesic polyarenes" by flash vacuum pyrolysis of adequately chosen starting materials (hydrocarbons or halogen derivatives) when many new C–C are being formed practically simultaneously by elimination of hydrogen or hydrogen halides [7–9].

We will denote the numbers of vertices, edges, and faces of polyhedra by V, E, and F, respectively. Euler's famous relationship $E = V + F + 2$ can be checked for all data presented in Table 1.

2 Complexity of Platonic and Archimedean polyhedra

The complexity of polyhedra can be judged either viewing them as graphs (presented as Schlegel diagrams) or as geometrical objects in the tridimensional space. The complexity of graphs corresponding to regular and semiregular polyhedra can be measured according to various topological criteria.

Two pairs of Platonic solids are dual, namely the cube with the octahedron, and the icosahedron with the dodecahedron, having pairwise the same number of edges, but being mutual duals of each other. Taking into account that the five regular polyhedra are named according to the number of their faces (the cube being synonymous with a hexahedron), one would expect these numbers to increase with the complexity, yet it is fairly evident that the "sphericity" increases according to their number V of vertices, and not their number F of faces. A simple measure of this sphericity is provided by the sum of planar angles meeting at each vertex. Evidently, the closer this sum is to 360°, the higher the sphericity. As seen in Table 1, the ordering of regular polyhedra according to this sum and to V is unambiguous: tetrahedron < octahedron < cube < icosahedron < dodecahedron, i. e. **1** < **2** < **3** < **4** < **5**.

However, the semiregular polyhedra cannot be ordered to these two simple criteria, because both of them present degeneracies (italics in Table 1). Not only dual truncated semiregular polyhedral pairs (Nos. **7** and **8**, as well as Nos. **14** and **15** in Table 1), but also other cases of degeneracy may be seen (italics in Table 1).

Several articles have discussed the complexity of Platonic polyhedra in the last decade, arriving at different conclusions about their ordering according to increasing complexity. Whereas Bonchev argued that the total walk count of the graphs agrees with the ordering according to F (**1** < **3** < **2** < **5** < **4**) [10]. Lukovits, Trinajstić, Nikolić and coworkers preferred the ordering according to the resistance distance and V (**1** < **2** < **3** < **4** < **5**), as well as to several topological indices of the corresponding graphs [11–13].

Only two papers have discussed the ordering and complexity of Archimedean polyhedra. The earliest by Balaban and Bonchev [14] examined several criteria, including the best of all, namely the solid angle at a vertex of the polyhedron, which is completely

Table 1 Data for Platonic and Archimedean polyhedra

No.	Platonic (regular) polyhedron	Faces meeting at a vertex	V	E	F	Sum of planar angles (°)	Solid angle (srad)
1	Tetrahedron	3 triangles	4	6	4	180	0.55
2	Octahedron	4 triangles	6	*12*	8	240	1.36
3	Cube	3 squares	8	*12*	6	270	1.57
4	Icosahedron	5 triangles	12	*30*	20	*300*	2.63
5	Dodecahedron	3 pentagons	20	*30*	12	324	2.96
No.	Archimedean (semiregular) polyhedron	Faces meeting at a vertex	V	E	F	Sum of planar angles (°)	Solid angle (srad)
6	Truncated tetrahedron	2 hexagons + 1 triangle	*12*	18	8	*300*	1.91
7	Cuboctahedron	2 squares + 2 triangles	*12*	24	*14*	*300*	2.47
8	Truncated cube	2 octagons + 1 triangle	*24*	*36*	*14*	*330*	2.80
9	Truncated octahedron	1 square + 2 hexagons	*24*	*36*	*14*	*330*	3.14
10	Small rhombicuboctahedron	1 triangle + 3 squares	*24*	48	*26*	*330*	3.48
11	Snub cube[a]	4 triangles + 1 square	*24*	*60*	38	*330*	3.59
12	Icosidodecahedron	2 triangles + 2 pentagons	30	*60*	*32*	336	3.67
13	Great rhombicuboctahedron	1 square + 1 hexagon + 1 octagon	48	72	*26*	345	3.95
14	Truncated dodecahedron	1 triangle + 2 decagons	*60*	*90*	*32*	*348*	3.87
15	Truncated icosahedron	1 pentagon + 2 hexagons	*60*	*90*	*32*	*348*	4.25
16	Small rhombicosidodecahedron	1 triangle + 2 squares + 1 pentagon	*60*	120	*62*	*348*	4.44
17	Snub dodecahedron[a]	4 triangles + 1 pentagon	*60*	150	92	*348*	4.51
18	Great rhombicosidodecahedron	1 square + 1 hexagon + 1 decagon	120	180	*62*	354	4.71

[a] Chiral polyhedron

devoid of degeneracy; however, at that time the data for the two snub semiregular polyhedra (**10** and **17**) were missing. Soon afterwards, Rajtmajer et al. [15] concluded that only a partial ordering was possible on the basis of several contradictory criteria, leaving thus seven out of the thirteen polyhedra as not-comparable. Their result is depicted as a Hasse diagram in Fig. 1, with numbers corresponding to names in Table 1.

In the present note these two missing data have been calculated, and are being displayed in Table 1. The ordering is now complete without any doubt or degeneracy,

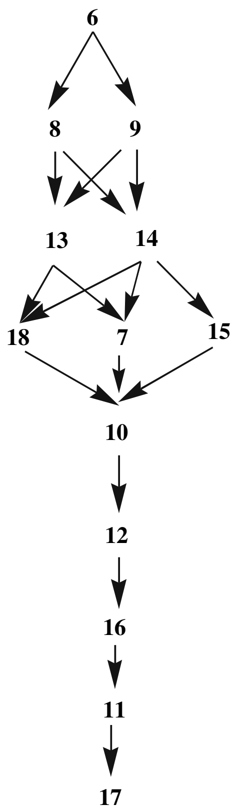

Fig. 1 Hasse diagram of the partial order for semiregular polyhedra (after Ref. [15])

according to solid angles, as follows (integrating regular polyhedra, marked by underlining, with semiregular polyhedra): **1** < **2** < **3** < **6** < **7** < **4** < **8** < **5** < **9** < **10** < **11** < **12** < **13** < **14** < **15** < **16** < **17** < **18**.

The solid angle, measured in steradians (srad), is defined as the ratio between the area of the sphere portion delimitated by planes with known planar angles meeting at the center of the sphere and the square of the sphere radius, r. Details about calculating solid angles for up to four planes [16] were provided earlier with explicit formulas for all polyhedra except **10** and **17** [17]. Evidently, since the total area of a sphere is $4\pi r^2$, the maximum solid angle is $2\pi = 6.28$ srad. It should be mentioned that whereas three planar angles (at the apex of a trigonal pyramid) define uniquely a solid angle, for planar angles as in a tetragonal or pentagonal pyramid with isosceles triangles having four or five edges meeting at the apex need more parameters for the characterization of the geometry.

The coordinates of the snub dodecahedron are given by Ref. [18].

$$(\pm 2\alpha, \pm 2, \pm 2\beta),$$
$$(\pm(\alpha + \beta/\tau + \tau), \pm(-\alpha\tau + \beta + 1/\tau), \pm(\alpha/\tau + \beta\tau - 1)),$$

$$(\pm(-\alpha/\tau + \beta\tau + 1), \pm(-\alpha + \beta/\tau - \tau), \pm(\alpha\tau + \beta - 1/\tau)),$$
$$(\pm(-\alpha/\tau + \beta\tau - 1), \pm(\alpha - \beta/\tau - \tau), \pm(\alpha\tau + \beta + 1/\tau)) \text{ and}$$
$$(\pm(\alpha + \beta/\tau - \tau), \pm(\alpha\tau - \beta + 1/\tau), \pm(\alpha/\tau + \beta\tau + 1)),$$

with an even number of plus signs, where

$$\alpha = \xi - 1/\xi$$

and

$$\beta = \xi\tau + \tau^2 + \tau/\xi,$$

where $\tau = (1 + \sqrt{5})/2$ is the golden mean and ξ is the real solution to $\xi^3 - 2\xi = \tau$, which is

$$\xi = \sqrt[3]{\frac{\tau}{2} + \frac{1}{2}\sqrt{\tau - \frac{5}{27}}} + \sqrt[3]{\frac{\tau}{2} - \frac{1}{2}\sqrt{\tau - \frac{5}{27}}}$$

or approximately 1.7155615.

Using the C++ program coordinates of all vertices $v_1, \ldots, v_{60}$ were calculated. Note that the minimal distance between two vertices is just the length of the edge of the snub dodecahedron. Using the computer, it was calculated that the minimal distance between two vertices is approximately 6.04373808. Using this, the adjacency matrix can be reconstructed. In this way it was found that neighbors of the distinguished vertex v_1 are v_4, v_{13}, v_{33}, v_{49} and v_{52}. Also there are four incidences between these neighbors. Namely, $v_{49}v_4$, v_4v_{52}, $v_{52}v_{33}$ and $v_{33}v_{13}$ are edges of the snub dodecahedron. Vectors $a_1 = v_1v_{49}, a_2 = v_1v_4, a_3 = v_1v_{52}, a_4 = v_1v_{33}$ and $a_5 = v_1v_{13}$ have been calculated. Denote by P_{ij} the plane defined by vectors i and j and denote $n_{i,j} = v_i \times v_j$. The dihedral angle between planes P_{ij} and P_{jk} is the angle between vertices n_{ji} and n_{jk}. It can be easily seen that the solid angle determined by these five vectors is the sum of solid angles defined by the following triplets of vectors:

(1) a_1, a_3 and a_5;
(2) a_3, a_4 and a_5;
(3) a_1, a_3 and a_5.

The solid angle of each of the triplets is calculated by using:

Girard's Theorem *Let α, β, γ be a solid with dihedral angles of three planes passing through the same point; then the solid angle defined by these planes is $\alpha + \beta + \gamma - \pi$.*

Summing up all three solid angles one gets 4.50969 srad. The results have been also verified by using the coordinates given in Ref. [18]. A similar procedure has been used for the snub cube (taking the data from [19] and [20]), and the solid angle has been calculated to be 3.59.

Morever, if one observes the five points adjacent to one vertex in either the snub cube or the snub dodecahedron, the following observations can be made. The volumes defined by four of these five points are zero (in fact the results are very close to zero and the discrepancy is due to the rounding of numbers). Using the fact that a tetragon can be inscribed in circle only if the sum of the opposite angles is equal to π, it can be checked that these points lie on the same circle (again, numerical data show a very small discrepancy due to rounding).

3 Conclusions

The solid angles of all Platonic and Archimedean polyhedra were calculated, and they allow an unambiguous non-degenerate ordering of all these 18 polyhedra, as displayed in Table 1 and in the sequence **1** < **2** < **3** < **6** < **7** < **4** < **8** < **5** < **9** < **10** < **11** < **12** < **13** < **14** < **15** < **16** < **17** < **18**. The lowest complexity is that of the tetrahedron **1**, and the highest complexity corresponds to the great rhombicosidodecahedron **18** (also called truncated icosidodecahedron).

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